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Boundary-induced drift of spirals in excitable media

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The motion of a spiral wave in an excitable medium due to interaction with a boundary is considered. The drift of the core and the frequency shift, evaluated analytically as a response to a small perturbation of the boundary, are found to be superexponentially weak. It is shown that for some range of parameters the spiral is unstable to small displacements away from the center of a circular domain.

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A great variety of experiments have revealed rotating spiral wave patterns in two-dimensional excitable media. These spirals arise as waves of oxidation in the Belousov-Zhabotinsky (BZ) reaction [1] and in the catalysis of CO on Pt substrates [2], as waves of electrical activity propagating along the axonal membrane of a neural cell [3], as temporally periodic and spatially organized contractions in muscular tissue [4], etc. The theory of wave propagation in excitable media can be described by the pair of "reaction-diffusion" equations [5,6]:

$$\partial_t u = \epsilon \nabla^2 u + \frac{f(u,v)}{\epsilon} ,$$
$$\partial_t v = \delta \epsilon \nabla^2 v + g(u,v) \ , \tag{1}$$

where $\epsilon \ll 1$ is a small positive parameter, u and v are the "fast" and "slow" variables, and $\delta = D_v / D_u$ is the ratio of the diffusion coefficients of the two variables. The properties of a particular system are given by the functions f and g. In the Fitzhugh-Nagumo (FN) model [7] $f = 3u - u^3 - v$, $g = u - \gamma v + \Delta$ with the parameters γ and Δ governing the kinetics of the medium.

In the one-dimensional (1D) case Eqs. (1) have a solution in the form of moving excited and quiescent zones, separated by *interfaces* of width $O(\epsilon)$ [5,6]. In the 2D case, Eqs. (1) reduce to a locally 1D problem $c_n = c(v) - \epsilon k$ where $c(v)$ is the interfacial velocity in the 1D case, c_n is normal velocity of the interface, and k is the local curvature of the interface [5,6]. For small ϵ, v deviates slightly from the stall value v_s, defined by $c(v_s) = 0$, and Eqs. (1) can be written in a universal form, independent of a particular model. This universality is reflected by a certain scaling of the system, due to Fife, [8] $v - v_s = \epsilon^{1/3} \tilde{v}$, $x = \epsilon^{2/3} \tilde{x}$, $t = \epsilon^{1/3} \tilde{t}$, $c(v_I) \approx \epsilon^{1/3} c_v \tilde{v}_I$, $\omega = \epsilon^{-1/3} \tilde{\omega}$, where $c_v \equiv dc(v)/dv|_{v=v_s}$ (in the FN model $v_s = 0$ and $c_v = -1/\sqrt{2}$).

We consider the case of small diffusion of the slow variable v, i.e., $\delta \ll 1$ [9,10]. In this case, the problem splits into an outer problem, where diffusion and the finite interfacial width can be ignored, and a small core region near the origin where this approximation breaks down. For the outer problem, in the frame corotating with the interface at the frequency ω, the Fife ansatz brings Eqs. (1) and the interfacial equation (after dropping tildes) to the form

$$\partial_t v^{\pm} - \omega \partial_\theta v^{\pm} = g^{\pm} \ , \tag{2}$$

$$c_n = c_v v - k \ , \tag{3}$$

where the signs "+" and "−" correspond to the excited and quiescent regions, respectively. $g^{\pm} \equiv g(u^{\pm}(v_s), v_s) = \text{const}$. A rigidly rotating spiral solution ($\partial_t = 0$) can be obtained both for the case of an infinite domain and for a finite circular domain. Spiral wave number selection is obtained by substituting the expressions for the normal velocity c_n and the curvature k into Eq. (3). One has the following equation [5,9,10] for the interfacial angle θ_I:

$$\frac{d\psi}{d\rho} = \left(\rho - \frac{\psi}{\rho} \right)(1+\psi^2) - B(1+\psi^2)^{3/2} \ , \tag{4}$$

where $\rho = \sqrt{\omega} r$, $\psi(\rho) \equiv \rho [d\theta_I(\rho)/d\rho]$. Equation (4) completed by the boundary condition is a nonlinear eigenvalue problem for B, related to ω by the relation $B = (c_v g^+ g^- \pi)/[\omega^{3/2}(g^+ - g^-)]$. The solution of the infinite domain problem $\psi_0(\rho)$ has the asymptotics $\psi_0(\rho) \propto -\rho/B$ at $\rho \to \infty$ and $\psi_0(0) = 0$, and the corresponding eigenvalue $B_0 \approx 1.738$ [5,9,10]. For the finite circular domain case, one has a correction to B_0, which can be obtained by means of matched asymptotics. Confinement of a spiral by a boundary results in the appearance of a thin boundary layer, where the solution is strongly perturbed, whereas in the rest of the domain it can be treated by the linearization of Eq. (4). The boundary layer is described by solving Eq. (4) assuming $R - \rho \ll R$. The problem is solved for no-flux ($\psi = 0$) boundary conditions at the radius R, i.e., that the interface approaches the boundary along its normal. Rigid rotation exists only if the spiral tip is exactly at the center of the domain. The matching yields the following correction $\delta\omega$ to the frequency of rotation in the infinite domain:

$$\delta\omega = \frac{\omega}{\beta} R^{-1-(1/B_0^4)} \exp\left[-\frac{R^3}{3B_0} - \frac{R^2}{2B_0^2} + \eta R \right] \ , \tag{5}$$

where

$$\eta = \frac{\pi}{2B_0} + \frac{B_0}{2} - \frac{1}{B_0^3} \approx 1.58 \ , \quad \beta \approx 1.4735 \ .$$

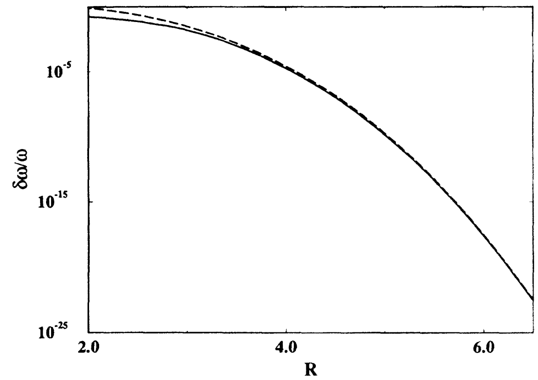

FIG. 1. The frequency shift $\delta\omega/\omega$ versus radius of the domain R. Solid line is the numerical solution of Eq. (4), dashed line is the analytical solution (5).

The superexponentially weak shift $\delta\omega$ is in good agreement with a numerical solution of Eq. (4) (see Fig. 1).

In a finite domain, rigid rotation is not generic. The interaction with the boundary (or other spirals) gives rise to both localized deformations, small in comparison with the spiral size, and "macroscopic" drifting of the spiral as a whole. In the following we derive the universal, in this case superexponential, asymptotics of spiral interaction with a perturbed circular boundary, if the distance from the spiral tip to the boundary is sufficiently large. This extremely weak interaction is the direct consequence of the absence of diffusion in the slow variable and the locality of Eq. (3) for $\epsilon \to 0$. We prove that this small drift can be both attractive or repulsive, depending on the domain radius. In the attractive case, this implies an instability of the rigidly rotating spiral.

For a slightly perturbed circular boundary (or, equivalently, for the case of a spiral tip displaced from the center of a circular domain), the boundary radius is given by $R(\theta)=R_0+\mu(\theta), \mu \ll R_0$. The no-flux boundary conditions here imply $\psi(R)=-\mu'/R$. We may eliminate the need for a fully nonlinear treatment at the boundary by replacing the boundary conditions at $R(\theta)$ by an effective boundary condition at R_0, which we take to be large. Expanding $\psi(R_0+\mu)$ around the ideal circular domain [i.e., $\psi_0(R_0)=0, \psi_0'(R_0)\approx R_0$] one obtains

$$\psi(R_0)=-R_0\mu-\frac{\mu'}{R_0}\approx -R_0\mu . \qquad (6)$$

The problem can be essentially solved within the linear approximation starting from the spiral solution in a circular domain. The key point of the calculation is that the perturbed boundary produces distortions in resonance with the translation modes of the free boundary problem. In order to overcome the secular growth of these modes in time (which is, in fact, a drift of the spiral as a whole), one needs to introduce a moving coordinate frame. The boundedness of the perturbation in that frame selects the unique value of the drift velocity.

In the system drifting with velocity $\mathbf{c_d}=(c_x,c_y)$ and rotating with frequency ω, Eq. (2) has the form

$$\partial_t v^{\pm}-\omega\partial_\theta v^{\pm}=g^{\pm}+\mathbf{c_d}\cdot\nabla v . \qquad (7)$$

To linear order in c_d, one may replace v in the right-hand side of Eq. (7) by v_0 in the corotating frame $v_0^{\pm}=(-g^{\pm}/\omega)[\theta-\theta^{\pm}(r)]+\text{const}$ with $\theta^+-\theta^-=\Delta\theta=2\pi g/(g^- - g^+)$. Without loss of generality, we may choose $g^+-g^-=1$. In our frame the perturbation induced by the steady drift is explicitly time dependent and can be written in the form

$$\mathbf{c_d}\cdot\nabla v_0=-\hat{C}\frac{g^{\pm}(i-\psi)}{2\rho}\exp[i(\omega t+\theta)]+\text{c.c.} , \qquad (8)$$

where $\hat{C}=(c_x-ic_y)/\sqrt{\omega}$. The translation modes of the infinite domain problem coincide exactly with Eq. (8) and have, respectively, eigenvalues $\pm i\omega$ [11]. Therefore steady drift is in resonance with the translation modes, and so can counteract the resonance induced by the boundary perturbations.

To find the value of $\hat{C}$ one has to determine the (linear) response of v to the drift in Eq. (7). As the drift perturbation is the sum of two complex-conjugate terms, the response to linear order similarly decomposes, and in the following we will only explicitly treat the $+i\omega t$ component. Solving Eq. (7) for the linear response of the v field and applying the continuity conditions for v at the shifted interface (as in [11]) yields the result

$$\delta\theta^+=\delta\theta^-\exp[i\Delta\theta] . \qquad (9)$$

where $\delta\theta^{\pm}\exp[i\omega t]$ are the perturbations to the interfaces. The linearized Eq. (3) then yields a closed equation for $\delta\theta^+$

$$\frac{\partial_\rho^2\delta\theta^+ +\left(\frac{2}{\rho}+\rho\psi-\frac{3\psi\psi'}{1+\psi^2}\right)\partial_\rho\delta\theta^+}{1+\psi^2}-i\delta\theta^+=\hat{C}f(\rho) , \qquad (10)$$

where

$$f(\rho)=\frac{i-\psi}{2\rho}\left[1+\frac{B}{\rho}\sqrt{1+\psi^2}\right]\exp[i\theta^+] . \qquad (11)$$

Equation (10) can be solved analytically exploiting our knowledge of the homogeneous solutions. One homogeneous solution is given by the translation eigenmode $\chi_1=(i-\psi/\rho)\exp[i\theta^+]$ and the second solution can be obtained via reduction of order $\chi_2=\chi_1\int_0^\rho dr'[W(r')/\chi_1^2]$, where the Wronskian $W(\rho)=\exp(-\int_0^\rho dr' r'\psi)\rho^{-2}(1+\psi^2)^{3/2}$. These solutions have the following asymptotic behaviors: $\chi_1\to 1/\rho, \chi_2\to\text{const}$ for $\rho\to 0$ and χ_1 is bounded, $\chi_2\sim\exp[\rho^3/(3B)]$ for $\rho\gg 1$. The general solution of Eq. (10) can be represented in the form

$$\delta\theta^+=A_1\chi_1+A_2\chi_2+\hat{C}Z(\rho) , \qquad (12)$$

where $Z(\rho)$ is the inhomogeneous solution which behaves as $(iB/2)\ln(\rho)+z_0$ as $\rho\to 0$, for some constant z_0, and is bounded for large ρ. The constants $A_{1,2}$ are fixed by the boundary conditions at $\rho=R_0$ and at the core $\rho\to 0$.

In order to satisfy the boundary conditions on the outer boundary we have to extract the resonant contributions from the perturbations given by Eq. (6). In fact, one needs only the first harmonics $\mu_1=\langle\mu(\theta+\omega t)\exp[-i\omega t]\rangle$. Then the bound-

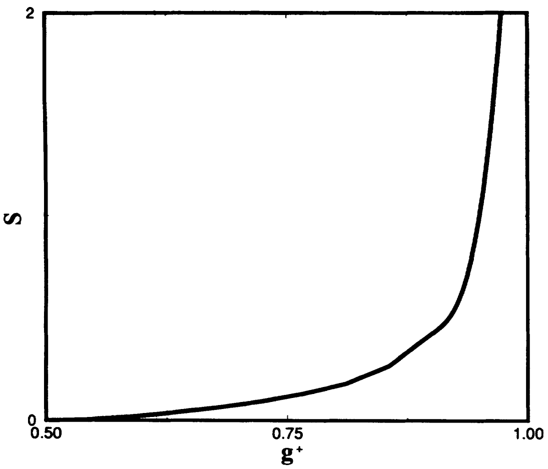

FIG. 2. Dependence $S=|S_2^+|(2/\sqrt{\omega})(-c_v/\pi)^{1/3}$ versus g^+.

ary conditions read as $\psi^+=R\partial_\rho\delta\theta^+=-\mu_1 R+\text{c.c.}$ which, using the fact that $\delta\theta^+\rightarrow A_2\chi_2$ for $\rho\gg 1$, immediately yields the following relation: $A_2\chi_2'(R_0)=-\mu_1$.

An additional relation between $\hat{C}$, A_1, and A_2 may be obtained by satisfying the regularity conditions at the core, where the outer equations (2) and (3) break down [9,11]. This is most simply done for the case $\epsilon^{1/3}\ll\delta^{1/3}\ll 1$ (so-called Bernoff core) where the problem still possesses sharp interfaces even in the core region. Thus we need to perform the "Bernoff scaling" in addition to the "Fife scaling" [11,9,10]: $\tilde{r}=r\delta^{-1/3},\tilde{v}=v\delta^{1/3}$. In terms of the Bernoff variables, the small ρ outer solution has the following asymptotics:

$$\delta\theta^+=iA_1e^{i\Delta\theta/2}\left(\frac{1}{\rho}-iB\right)+A_2\chi_2(0)+\hat{C}\left(z_0+\frac{iB}{2}\ln(\rho)\right)+O(\rho)\;, \tag{13}$$

which we must match to the large r core solution. The values of the constants $A_{1,2}$ are determined by this matching. If $c_d\sim\delta^{2/3}$, the drifting core problem can be treated in linear approximation. Following along the lines of [11], and defining $\tilde{C}\equiv\hat{C}\delta^{-2/3}$, we may obtain the linear response of the interface to the drift. Asymptotically, this has the form

$$\delta\theta^{\pm}=\tilde{C}\left(\frac{S_1^{\pm}}{\sqrt{\omega r}}+S_2^{\pm}+\cdots\right)\;. \tag{14}$$

The asymptotic constants $S_{1,2}^{\pm}$ are determined numerically from the solution of the perturbed core. In addition, one can show that $\delta\theta^+=-(\tilde{C}/\tilde{C}^*)\delta\theta^{-*}$. Together with Eq. (9), which also holds for $r\rightarrow\infty$, this implies $\arg(S_2^+)=\Delta\theta/2-\pi/2$. The dependence $|S_2^+|$ versus g^+ is given in Fig. 2. Note that $|S_2^+|$ vanishes for the symmetric spiral ($g^+=1/2$) and diverges for $g^+=0,1$. It follows from the symmetry of the core solution under $g^+\leftrightarrow 1-g^+$, that $S_2^+(g^+)=-S_2^+(1-g^+)$.

Comparing Eqs. (13) and (14) one achieves the matching if the relations $A_1=-i\delta^{-1/3}S_1^+\hat{C}\exp(-i\Delta\theta/2)$ and

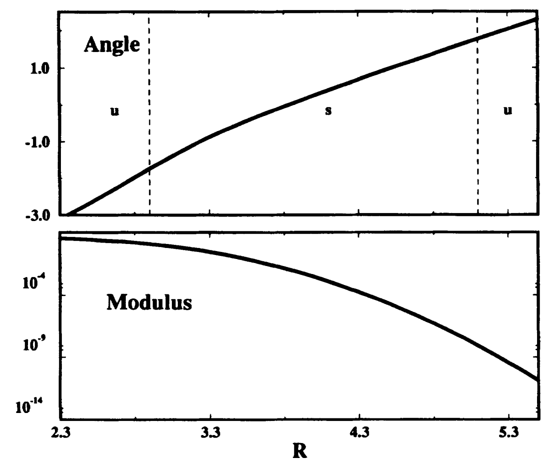

FIG. 3. The angle of the drift and the normalized velocity $|\hat{C}\delta^{-2/3}S_2^+/\mu|$ versus R for $g^+=0.7$. The dashed lines designate the stability zones of the spiral, given by the condition that the angle is between $-\pi/2$ and $\pi/2$. For example, for the distance $R\leqslant 2.9$ or $R\geqslant 5.1$ the central position is unstable.

$A_2=\delta^{-2/3}S_2^+\hat{C}/\chi_2(0)$ hold. The matching with the $\ln(\rho)$ term is achieved at higher order in $\delta^{1/3}$. Therefore the drift is given by the expression

$$\hat{C}=-\frac{\chi_2(0)\delta^{2/3}\mu_1}{\chi_2'(R_0)S_2^+}\propto\frac{\delta^{2/3}\mu_1}{S_2^+}\exp\left[-\frac{R_0^3}{3B}-i\frac{R_0}{B}\right]\;. \tag{15}$$

Equation (15) reveals the superexponentially weak interaction with the boundary. The direction of the drift actually also depends on the radius R_0 (see Fig. 3). Formally speaking, the drift vanishes as $\delta\rightarrow 0$. In this case the Bernoff scaling fails, and one needs to analyze the diffusionless core [12]. An analysis similar to the above shows that in this case the drift is proportional to $\epsilon^{2/3}$.

For the particular case of $\mu(\theta)$ given by a spiral displaced from the center of the circular boundary at the position (x,y), the perturbation is given by $\mu_1=\hat{x}/2+O(1/R_0)$ where $\hat{x}=x-iy$. Using the relation $\hat{C}\sim\partial_t\hat{x}$, one has the following equation of the motion:

$$\partial_t\hat{x}\propto\frac{\delta^{2/3}}{S_2^+}\exp\left[-\frac{R_0^3}{3B}-2i\frac{R_0}{B}\right]\hat{x}\;. \tag{16}$$

From Eq. (16) one sees that, depending on the radius of the boundary, the spiral either returns back to the center, or drifts toward the boundary. In fact, the stability characteristic changes periodically as a function of R_0.

The problem of the spiral motion in a circular domain can also be treated as a linear stability problem. The drift is then considered as a small shift of the eigenvalues corresponding to the translation modes due to confinement of the spiral. Then one recovers Eq. (16) as the expression for the shift of the eigenvalues.

We have obtained a quite unexpected result—the spiral interacts extremely weakly with the boundary. The drift turns out to be of order $\exp[-R^3]$. In oscillatory media the interaction is known to decay exponentially due to the screening effects from the emitted waves [13]. One may expect an

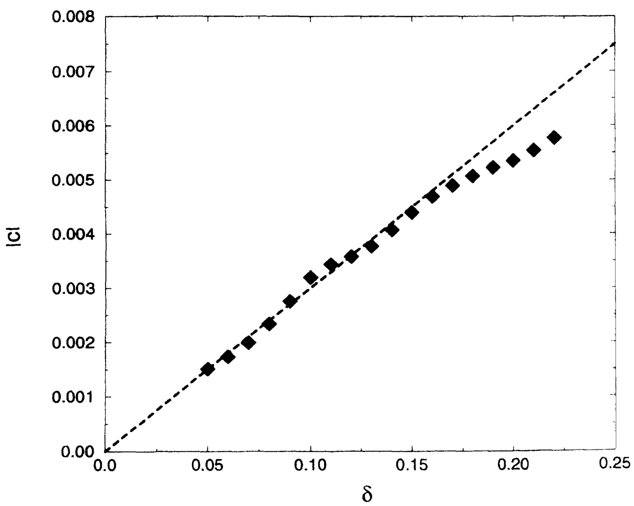

FIG. 4. Dependence $|c|$ versus δ. The parameters of the simulations with the model of Ref. [21] are $a=0.75$, $b=0.01$, $\epsilon=0.002$, the domain size 8×8, number of the grid points 141×141.

asymptotically exponential screening of the interaction in the case of significant diffusivity of the slow variable. Also, it should be noted that the Bernoff limit, while well defined mathematically, is hard to acheive experimentally, as it requires that $\epsilon^{1/3} \ll \delta^{1/3} \ll 1$, which means that both ϵ and δ must be *exceedingly* small, say $\delta \sim 10^{-5}$, $\epsilon \sim 10^{-8}$. However, we find from our simulations that the extremely weak interaction with the boundary derived herein is valid even outside the Bernoff limit. The drift is however linear in δ, as opposed to $\delta^{2/3}$ in this case (see Fig. 4). For the diffusionless core, we expect the drift to be proportional to ϵ in typical situations. This is consistent with the analysis of Keener [14] outlining the violation of Fife scaling for the not-too-small values of $\epsilon \sim 10^{-2}$ encountered in nature.

Another noteworthy result of our calculation is that the confinement of the spiral into a finite region always increases the frequency. One expects, therefore, that in the multispiral case, spirals will coexist, and the symmetry breaking analogous to that in Ref. [15] will not develop. This statement is in qualitative agreement with numerical simulations [10,16], although one cannot exclude a symmetry breaking for not very small ϵ, δ.

One remaining problem is the interaction with the plane boundary. Formally speaking, the presented method fails to describe such an interaction, because μ is not small any more. Moreover, one has the effect of the tearing off of the interfaces at the boundary. However, our analysis shows that the influence of remote parts of the interfaces is negligibly weak, and the main contribution comes from the parts closest to the boundary, where the variations of μ are relatively small. In this way we obtain an $\sim \exp[-X^3/3B - 2iX/B]$ interaction with the plane boundary at the distance X. This type of interaction exhibits bound states at the distances $X \sim \pi B n$, which have indeed been seen in numerical simulations [17]. Similarly, the extremely local nature of the interaction with the boundary accounts qualitatively for the results of the numerical simulations [18] of the interaction of the spiral (in a medium with no slow field diffusion) with a defect, where it was found that the interaction was only effective up to a distance of order of the core radius.

In recent papers the interaction of the spiral with the boundary has been considered within the framework of a phenomenological kinematic approach. Although qualitatively the results look reasonable, the answers exaggerate the effects of the interaction ($1/R$ in [19] and exponential in [20]).

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